

Computing Cohomology Rings in Cubical Agda

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Abstract

In Homotopy Type Theory, cohomology theories are studied synthetically using higher inductive types and univalence. This paper extends previous developments by providing the first fully mechanized definition of cohomology rings. These rings may be defined as direct sums of cohomology groups together with a multiplication induced by the cup product, and can in many cases be characterized as quotients of multivariate polynomial rings. To this end, we introduce appropriate definitions of direct sums and graded rings, which we then use to define both cohomology rings and multivariate polynomial rings. Using this, we compute the cohomology rings of some classical spaces, such as the spheres and the Klein bottle. The formalization is constructive so that it can be used to do concrete computations, and it relies on the Cubical Agda system which natively supports higher inductive types and computational univalence.

Keywords: Synthetic Cohomology Theory, Cohomology Rings, Polynomials, Homotopy Type Theory

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1 Introduction

A fundamental idea in algebraic topology is that spaces can be analyzed in terms of homotopy invariants—functorial assignments of algebraic objects to spaces. Primary examples of such invariants include homotopy groups, homology groups and, the concept of study in this paper, cohomology groups and rings. Both homology and cohomology groups are often much easier to compute than homotopy groups, making them ubiquitous in modern pure mathematics as well as in applied subjects, like topological data analysis [8].

Intuitively, the cohomology groups $H^n(X, G)$ of a space X relative to an abelian group G , in particular \mathbb{Z} , characterize the connected components of X as well as its $(n + 1)$ -dimensional holes. Fig. 1 depicts the circle, S^1 , and two circles that have been glued together in a point, i.e. the *wedge sum* of two circles, $S^1 \vee S^1$. The fact that these spaces have a different number of holes is captured cohomologically by $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(S^1 \vee S^1, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, which geometrically means that they have one, respectively two, 2-dimensional holes (i.e. empty interiors). As cohomology groups are homotopy invariants, this then means that the spaces cannot be continuously deformed into each other.

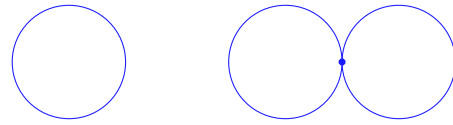


Figure 1. S^1 and $S^1 \vee S^1$.

More formally, one considers cohomology with coefficients in an abelian group G as a family of contravariant functors $H^n(_, G) : \text{Space} \rightarrow \text{AbGroup}$ assigning to a space¹ X its n th cohomology group with G -coefficients. This assignment is assumed to satisfy the Eilenberg-Steenrod axioms [14] and be stable under weak homotopy equivalence. This is a precise mathematical way to say that the functors $H^n(_, G)$ identify spaces of the same “shape”. Such a family of functors is then said to constitute a *cohomology theory*.

Classically, homology groups have a definition very similar to that of cohomology groups and can sometimes be easier to compute, but cohomology groups (when taken with coefficients in a ring R) have an important advantage: they can be equipped with a graded product

$$\smile : H^n(X, R) \rightarrow H^m(X, R) \rightarrow H^{n+m}(X, R)$$

This product, known as the *cup product*, can then be used to construct a graded ring $H^*(X, R)$, known as the *cohomology ring* of X with coefficients in R . This gives an even more fine-grained invariant than just the (co)homology groups, as it can help to distinguish spaces for which all (co)homology groups agree. As an example, consider Fig. 2 which depicts

¹We are intentionally vague about what a “space” is, but it can for example be a topological space or some more combinatorial presentation like simplicial or CW complexes, or as in HoTT/UF, a type.

a torus and the “Mickey Mouse space” consisting of a sphere glued together with two circles.

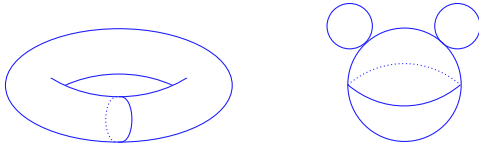


Figure 2. \mathbb{T}^2 and $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$.

Let X be either of the two spaces in the figure; one can prove that $H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ as they are connected, $H^1(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ as they each have two 2-dimensional holes (the circle in the middle of the torus and the one going around the interior, and the “ears” of Mickey), and $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ as they have one 3-dimensional hole each (the interior of the torus and Mickey’s head). Furthermore, their higher cohomology groups are all trivial as they do not have any further higher dimensional cells. The cohomology groups are hence insufficient to tell the spaces apart. However, one can show that the cup product on $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$ is trivial, while for \mathbb{T}^2 it is not. So $H^*(\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1, \mathbb{Z}) \not\cong H^*(\mathbb{T}^2, \mathbb{Z})$, which again means that the spaces cannot be continuously deformed into each other.

Providing a suitable framework for formalizing the graded ring structure of the cohomology rings of a space is one of the main goals of this paper. The formalization is carried out in *Cubical Agda* [32], an extension of *Agda* [28], with native support for higher inductive types (HITs) and computational univalence. This is a direct implementation of Homotopy Type Theory and Univalent Foundations (HoTT/UF), where the idea that equalities between terms of a type can be represented as paths in a space is taken very literally. This makes it possible to develop homotopy theory *synthetically* as in the HoTT Book [30] and many classical results from homotopy theory have been formalized this way. Synthetic cohomology theory in Book HoTT was initially studied at the IAS special year on HoTT/UF in 2012–2013 [27] and has since been used to develop the Eilenberg–Steenrod axioms [9], cellular cohomology [6], Atiyah–Hirzebruch and Serre spectral sequences [31], and to prove that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$ [4]. By developing these results cubically in *Cubical Agda*, many proofs can be substantially simplified, as illustrated for synthetic homotopy theory by Mörtberg and Pujet [26] and for integral cohomology theory by Brunerie, Ljungström, and Mörtberg [5]. This is made possible by the fact that univalence computes, and more importantly, that all computation rules for HITs hold strictly and not just up to paths as in Book HoTT.

This present paper is a continuation of the work by Brunerie et al. [5], where the integral cohomology groups $H^n(X, \mathbb{Z})$ were developed and studied in *Cubical Agda*. There, the

authors computed multiple \mathbb{Z} -cohomology groups of various spaces represented as HITs, including the spheres, torus, real/complex projective planes, and Klein bottle. They also gave new synthetic constructions of the group structure on $H^n(X, \mathbb{Z})$ and of the cup product, which led to substantially simplified proofs compared to those of Brunerie [4]. As all the proofs were constructive, *Cubical Agda* could be used to compute with these cohomology operations. The present paper is also carefully written so that all proofs are constructive and can be used to do concrete computations. Having the possibility of doing proofs simply by computation is one of the most appealing aspects of developing synthetic cohomology theory cubically. As this is not possible with pen and paper proofs, or even with many formalized proofs in Book HoTT, one often has to resort to doing long calculations by hand. If proofs instead can be carried out using a computer, many of these long calculations become obsolete.

Contributions. The main contribution of the paper is the first fully formalized synthetic definition of the cohomology ring $H^*(X, R)$ given a space X and ring R in Section 5. While previous authors have discussed synthetic constructions of the cup product [2, 4, 5], we lift it to a well-chosen construction of $H^*(X, R)$. Furthermore, we compute some of these rings for various X and R , including the spheres, real and complex planes, Klein bottle, and various wedges of these spaces. These cohomology rings are all equivalent to some $R[X_1, \dots, X_n]/I$, i.e. a (multivariate) polynomial ring modulo an ideal of relations. These particular examples of spaces are chosen because they illustrate different aspects of the computations as well as the need to sometimes change the ring R in order to distinguish spaces.

In order to construct $H^*(X, R)$, we give a general formalization of graded rings (Section 3), which in turn requires us to formalize direct sums of \mathbb{N} -indexed families of groups (Section 3.1). This is interesting on its own, as special care has to be taken to remain constructive while not imposing undesirable decidability assumptions. To facilitate convenient formal proofs involving the direct sum, we give a new definition of it using a HIT. As a byproduct, we get a new definition of (multivariate) polynomial rings (Section 4) which is well-suited both for programming and proving. As part of this, we rely on the structure identity principle (SIP)—univalence for *structured* types—to transport proofs between different representations of the direct sum (Section 3.2.1).

All results in the paper have been formalized in *Cubical Agda* and is part of the *agda/cubical* library (available at <https://github.com/agda/cubical/>). The code in the paper is mainly literal *Agda* code taken verbatim from the library, but we have taken some liberties when typesetting, e.g. shortening notations and omitting some universe levels. The connections between the paper and formalization is summarized in a summary file which can be found on [GitHub](https://github.com). The

summary file typechecks with Agda's `--safe` flag, which ensures that there are no admitted goals or postulates.

2 Background

Here we briefly overview prior work and survey the fundamental concepts underpinning the results of the paper.

2.1 Homotopy Type Theory in Cubical Agda

Agda is a dependently typed functional programming language, in which each program may be interpreted as a proof in intensional type theory [24]. For a brief overview of the syntax, see Table 1. With the addition of the univalence ax-

Concept	Syntax
Function types	$A \rightarrow B$
Dependent function types	$(x : A) \rightarrow B$
Implicit dep. function types	$\{x : A\} \rightarrow B$
Function application, $f(x)$	$f\ x$
Dependent pair types	$\Sigma A B$ and $\Sigma[x \in A] B\ x$
Universes (at level ℓ)	$\text{Type } \ell$

Table 1. Overview of Agda syntax

iom of Voevodsky [33], which says that equivalence of types can be promoted to equalities/paths of types, Agda may be used as a proof assistant for various flavors of HoTT/UF. Cubical Agda is an extension of Agda for a particular flavor of HoTT/UF – the cubical type theory of Cohen et al. [10] and Coquand et al. [12]. In Cubical Agda, unlike plain Agda, univalence is given computational content, allowing us to carry out computations involving it. Cubical Agda also has native support for HITs. These types can, in particular, be used to define quotient types and various spaces.

The major difference when working in Cubical Agda compared to vanilla Agda or Book HoTT is that the primary identity type over a type A is changed from Martin-Löf's inductive construction [23] to a primitive *path*-type. The identification $x \equiv y$ is captured by $\text{Path } A\ x\ y$, the type of functions $p : \mathbb{I} \rightarrow A$, where \mathbb{I} is a primitive interval type, restricting definitionally to x and y at the endpoints $i0$ and $i1$ of \mathbb{I} . The interval also comes with join, meet and inversion operations $_ \wedge _$, $_ \vee _$, $_ \sim _$, endowing it with the structure of a De Morgan algebra.

For an example of the cubical path type in action, consider the following proof of function extensionality:

```
funExt : {f g : A → B} → ((x : A) → f x ≡ g x) → f ≡ g
funExt p i = λ x → p x i
```

Above, the term $p\ x$ is interpreted as a path $\mathbb{I} \rightarrow B$, restricting definitionally to $f\ x$ at $i0$ and to $g\ x$ at $i1$. Exploiting the η -rule for function types, the above restricts definitionally to f at $i0$ and to g at $i1$.

Cubical Agda also has a dependent path type, PathP . Given a line of types $A : \mathbb{I} \rightarrow \text{Type}$ (which we informally may think of as $A\ i0 \equiv A\ i1$) and points $x : A\ i0$, $y : A\ i1$, the type $\text{PathP } A\ x\ y$ expresses that x and y may be identified relative to A . The regular (homogeneous) path type $_ \equiv _$ is, by definition, $\text{PathP } (\lambda i \rightarrow A)$, i.e. the special case when the line of types is constant. For an example of how PathP is used in Cubical Agda, consider the characterization of equality in Σ -types: given (x, bx) , $(y, by) : \Sigma A B$, a path $p : x \equiv y$ and a dependent path $\text{PathP } (\lambda i \rightarrow B (p\ i))\ bx\ by$, we get a path of pairs $(x, bx) \equiv (y, by)$. In Cubical Agda:

```
Σ≡ : {x y : A} {bx : B x} {by : B y} (p : x ≡ y)
  → PathP (λ i → B (p i)) bx by → (x , bx) ≡ (y , by)
Σ≡ p q = λ i → (p i , q i)
```

As previously mentioned, Cubical Agda also supports HITs. In particular, this allows us to construct various (homotopy types of) topological spaces. For a simple example, consider the HIT describing the circle, defined by a point `base` and the identification `loop : base ≡ base`.

```
data S1 : Type where
  base : S1
  loop : base ≡ base
```

HITs automatically come equipped with elimination principles. For instance, in order to define a dependent function $(x : S^1) \rightarrow B\ x$, we need to provide a point $b : B\ \text{base}$ and, given $i : \mathbb{I}$, a point p_i reducing definitionally to b when $i = i0$ and $i = i1$, i.e. a dependent path $\text{PathP } (\lambda i \rightarrow B (\text{loop } i))\ b\ b$. In Cubical Agda, definitions involving HITs can also be written by pattern-matching. For a simple example, consider the definition of the inversion map on S^1 .

```
invS1 : S1 → S1
invS1 base = base
invS1 (loop i) = loop (~ i)
```

Apart from topological spaces, HITs can also be used to capture important type operations, such as truncations. For instance, the propositional truncation is defined by:

```
data ||_||-1 (A : Type ℓ) : Type ℓ where
  |_: A → || A ||-1
  squash : (x y : || A ||-1) → x ≡ y
```

This HIT takes a type A and forces it to be a *proposition*, or (-1) -type: a type where any two points are identified up to a path. This is an important construction for capturing logical propositions in HoTT/UF. Indeed, we wish to think of logical propositions as having at most one witness and not any higher mathematical structure. For instance, we define existential quantification as:

$$\exists [a \in A] (P a) = || \Sigma A P ||_{-1}$$

In this paper, we follow the HoTT Book terminology and say that a *merely* exists when it is existentially quantified.

Also, note that the propositional truncation in the definition is crucial. In HoTT/UF ΣAP , is without the truncation interpreted as the total space of P , which may be highly non-trivial.

We can also move one step up and define set truncation:

```
data ||_||₀ (A : Type ℓ) : Type ℓ where
  |_| : A → || A ||₀
  squash : (x y : || A ||₀) (p q : x ≡ y) → p ≡ q
```

The set truncation turns any type into a *set*, or 0-type: a type where all path type are propositions. This notion allows us to capture the idea of a classical set: a set in HoTT/UF is simply a collection of points without any higher homotopical structure. We can iterate this procedure and define $||_||_n$ for any $n > 0$. The n -truncation $||A||_n$ turns A into an n -type: a type over which all path types are $(n - 1)$ -types. For the general definition of $||_||_n$ one has to use a slightly different approach than we have for $||_||_{-1}$ and $||_||_0$ based on the hub-and-spoke construction [30, Chapter 7.3].

Later on, we will see that HITs also play an important role in the construction of types quotiented by a relation (as opposed to topological spaces). The idea is that we may add paths between points to capture the relation we want to quotient by and then set truncate to kill off higher homotopical structure arising from the imposed relations.

Finally, we will need pointed types. A pointed type is simply a pair (A, a_0) where A is a type and $a_0 : A$ is a chosen basepoint. In Cubical Agda, we define the universe of pointed types by:

$$\mathbf{Pointed} = \Sigma [A \in \mathbf{Type}] A$$

Given a pointed type A , we write $\mathbf{typ} A$ for the underlying type and $\mathbf{pt} A$ for the basepoint. We will, however, often abuse notation and simply write A for the pointed type (A, a_0) .

2.2 The structure identity principle

It is often the case that the types we are interested in come equipped with some structure on it. For instance, a group consists of an underlying set G with a group structure on G :

$$\mathbf{GroupStr} G = \sum_{(1, \cdot, _^{-1}) : \mathbf{RawGroupStr} G} \mathbf{GroupAx} (G, 1, \cdot, _^{-1})$$

Above, $\mathbf{RawGroupStr}(G)$ expresses that G has a raw group structure consisting of a neutral element and the usual group operations, whereas $\mathbf{GroupAx}$ expresses that it satisfies the group axioms. Note that $\mathbf{GroupAx}$ is a proposition as G is a set. Many other algebraic structures can be captured in this way, e.g. monoids, rings, fields, etc. A reasonable question to ask in a univalent setting is whether an equivalence of types can be promoted to an equality of structured types, such as groups. The *Structure Identity Principle* (SIP) [30, Section 9.8] is an informal principle which attempts to answer

this: given two structured types (A, S_A) and (B, S_B) and an equivalence of underlying types $A \simeq B$ which is a homomorphism with respect to the structure in question, we get a path of structured types $(A, S_A) \equiv (B, S_B)$. For instance, an isomorphism of groups G and H induces a path $G \equiv H$. This form of invariance up to isomorphism is one of the big strengths of univalent mathematics as it lets programs and proofs be transported between isomorphic structured types.

Perhaps more importantly, the SIP may also be used to transfer proofs between (partially defined) structures. Suppose, for instance, that we are given a group G and type H with just a raw group structure $(1_H, \cdot_H, _^{-1H})$. While it may be difficult to show directly that this raw group structure respects the group axioms, it may be easier to show that the equivalence $\varphi : G \simeq H$ of types preserves the raw structure from G to H .² The SIP then gives an induced group structure on H which is path-equal, as groups, to G . Thus, the SIP allows us to transport proofs between structured types in an efficient way. This has been implemented in the Cubical Agda standard library for most algebraic structures using the cubical SIP of Angiuli, Cavallo, Mörtberg and Zeuner [1].

2.3 Cohomology theory in Cubical Agda

For the construction of cohomology in HoTT/UF, we use Brown representability [3] and define cohomology groups as homotopy classes of maps into Eilenberg-MacLane spaces. Classically this is provably equivalent to the singular cohomology of the spaces we consider [18]—but we take it as our definition. Given an abelian group G , we define the n th cohomology group of a type X with G -coefficients by:

$$H^n(X, G) = || X \rightarrow K(G, n) ||_0$$

where $K(G, n)$ is the n th Eilenberg-MacLane space of G . Its construction as a HIT is well-known in HoTT/UF and is due to Licata and Finster [21]. One can construct maps:

$$\begin{aligned} +_k &: K(G, n) \rightarrow K(G, n) \rightarrow K(G, n) \\ -_k &: K(G, n) \rightarrow K(G, n) \end{aligned}$$

and, for a ring R :

$$\smile_k : K(R, n) \rightarrow K(R, m) \rightarrow K(R, n + m)$$

These operations induce, by post-composition, operations on cohomology groups, which we denote by $+_h$, $-_h$, \smile_h , respectively. Later on, we will see that these operations, by construction, induce the ring structure on $H^*(X, G)$.

The cup product, with ring laws, was first introduced (in HoTT) by Brunerie [4] for \mathbb{Z} -coefficients. These definitions and results, in their entirety, have, however, never been formally verified for Brunerie's original definition of the cup

²For groups it of course suffices to preserve only the \cdot operation for φ to be a group homomorphism, but this is a quite special property of groups and for general structures the whole raw structure needs to be preserved.

product (although *most* of them have by Baumann [2]). Nevertheless, in Brunerie et al. [5], the cup product was given a more concise, recursive definition, which resulted in a complete formalization of the graded ring axioms. The work in [5] was only concerned with \mathbb{Z} -coefficients however, but the authors emphasized that the results are easily generalized to hold for arbitrary coefficients. The cup product and ring axioms have since been formalized for arbitrary coefficients and can be found in the `agda/cubical` library. In particular, it is shown that:

- The element $0_h : H^n(X, R)$ defined by

$$0_h = |\lambda x \rightarrow 0_k|$$

where $0_k : K(R, n)$ is the basepoint, right- and left-annihilates. That is, for $x : H^m(X, R)$, we have

$$x \smile_h 0_h \equiv 0_h \smile_h x \equiv 0_h$$

- The element $1_h : H^0(X, R)$ defined by

$$1_h = |\lambda x \rightarrow 1_r|$$

is a multiplicative unit. So, for $x : H^n(X, R)$, we have

$$x \smile_h 1_h \equiv 1_h \smile_h x \equiv x$$

- The cup product is associative.
- The cup product left- and right-distributes over $+_h$.
- The cup product is graded-commutative. For $x : H^n(X, R)$ and $y : H^m(X, R)$, we have

$$x \smile_h y \equiv (-1)^{nm}(y \smile_h x)$$

In particular, when $R = \mathbb{Z}_2$ (i.e. $\mathbb{Z}/2\mathbb{Z}$), we have commutativity, since in this case inversion is given by the identity map.³

We note that the above identities, in most cases, hold up to a dependent path. For instance, right-annihilation holds up to a path in \mathbb{N} identifying $n + 0$ with n .

3 Formalizing graded rings

Both cohomology rings and polynomials rings are *graded* rings. This means that their underlying additive groups can be decomposed as a direct sum $\bigoplus_{i:\mathbb{N}} G_i$ of abelian groups G_i with a graded multiplicative operation $\star : G_i \rightarrow G_j \rightarrow G_{i+j}$. For \star to be well-defined, we need the indexing set I to be a monoid $(I, e, +)$; for both polynomials and cohomology rings, we pick $(\mathbb{N}, 0, +_)$. Indeed, the polynomial ring $R[X]$ is trivially graded by \mathbb{N} if we let G_i constantly be R and \star the multiplication of R . The cohomology ring $H^*(X, R)$, on the other hand, is defined as $\bigoplus_{i:\mathbb{N}} H^i(X, R)$ together with the cup product.

In this section we study how these notions can be formalized constructively in HoTT while avoiding decidability

³Graded commutativity for arbitrary coefficients was first formalized by Baumann [2] and is currently being developed in the `agda/cubical` library.

assumptions. The reason not to restrict to types with decidable equality is to support $\mathbb{R}[X]$ and $\mathbb{C}[X]$ as well as spaces for which the cohomology groups do not have decidable equality. Furthermore, as we want to be able to do concrete computations, we are careful to ensure that everything is constructive. To achieve this, we crucially rely on higher inductive types to define well-behaved quotient types.

3.1 Direct sums

To formalize graded rings we first need to formalize the direct sum of a family of abelian groups over an arbitrary (not necessarily monoidal) indexing set I . Given abelian groups G_i indexed by a set I , the direct sum is often defined as

$$\bigoplus_{i:I} G_i := \{(g_i)_{i \in I} \mid \exists \text{ finite } J \subset I, \forall n \notin J, g_n = 0 \in G_n\}$$

This definition can be adapted to HoTT by taking dependent functions $g : (i : I) \rightarrow G_i$ for which there *merely* exists a finite subset J of I such that for all $j \notin J$ we have $g_j \equiv 0$. To make this precise we need a notion of finite subsets together with a membership relation. Various approaches to finite sets in HoTT were studied by Frumin, Geuvers, Gondelman and van der Weide [15], but as is common in constructive mathematics, the notions bifurcate into constructively distinct, but classically equivalent, notions. For our purposes, the most suitable notion seems to be Kurotowski finite sets where finite subsets can be represented as duplicate-free lists of elements [15, Theorem 2.8]. However, as remarked above, for cohomology and polynomials rings we do not need general direct sums, but only those indexed by \mathbb{N} . In this case, the definition can be simplified as

$$\bigoplus_{i:\mathbb{N}} G_i := \{(g_i)_{i \in \mathbb{N}} \mid \exists k \in \mathbb{N} \forall n > k, g_n = 0 \in G_n\}$$

This simplification is possible as \mathbb{N} has a total order with finite initial segments. Indeed, initial segments are finite subsets and all finite subsets are included in an initial segment by computing the maximum of the subset. This definition is easy to formalize in Cubical Agda:

```

⊕Fun : (G : ℕ → AbGroup) → Type
⊕Fun G = Σ[ g ∈ ((n : ℕ) → ⟨ G n ⟩) ]
  ∃[ k ∈ ℕ ] ∀ n → n > k → g n ≡ 0⟨ G n ⟩
    
```

Here $\langle G n \rangle$ is the underlying type and $0\langle G n \rangle$ is the zero of $G n$. To prove that this is an abelian group we lift the operations pointwise, e.g. addition can be defined as:

```

_+⊕Fun_ : ⊕Fun G → ⊕Fun G → ⊕Fun G
(f , pf) +⊕Fun (g , pg) = ((λ n → f n +⟨ G n ⟩ g n) , prf)
  where
  prf : ∃[ k ∈ ℕ ] ∀ n → n > k → f n +⟨ G n ⟩ g n ≡ 0⟨ G n ⟩
    
```

We omit the proof of `prf`, but it is easily proved using the fact that its type is a proposition which means that the eliminator of propositional truncation lets us extract the points at which f and g become zero from `pf` and `pg`.

As the second component in $\oplus\text{Fun}$ is propositionally truncated, it suffices to prove that the underlying functions are equal in order to show that two elements of $\oplus\text{Fun}$ are equal:

```

 $\oplus\text{Fun} \equiv \{x\ y : \oplus\text{Fun}\ G\} \rightarrow \text{fst}\ x \equiv \text{fst}\ y \rightarrow x \equiv y$ 
 $\oplus\text{Fun} \equiv \Sigma \equiv \text{Prop}\ (\lambda \_ \rightarrow \text{squash})$ 

```

As the operations are defined pointwise, the fact that $\oplus\text{Fun}$ is an abelian group is then easily proved using function extensionality. For example, commutativity is proved as follows:

```

 $+\oplus\text{FunComm} : (x\ y : \oplus\text{Fun}\ G) \rightarrow x\ +\oplus\text{Fun}\ y \equiv y\ +\oplus\text{Fun}\ x$ 
 $+\oplus\text{FunComm}\ (f\ ,\ \_)\ (g\ ,\ \_) =$ 
 $\oplus\text{Fun} \equiv (\text{funExt}\ (\lambda\ n \rightarrow +\text{Comm}\langle G\ n\rangle\ (f\ n)\ (g\ n)))$ 

```

While $\oplus\text{Fun}$ is convenient for proving the group laws, it is not very convenient for the multiplicative structure of the graded rings, as we will see in the next section. To remedy this, we have also formalized an equivalent definition using a HIT. This definition works just as well for arbitrary indexing types I , so we consider the following general form:

```

data  $\oplus\text{HIT}\ (I : \text{Type})\ (G : I \rightarrow \text{AbGroup}) : \text{Type}\ \text{where}$ 
   $0\oplus : \oplus\text{HIT}\ I\ G$ 
   $\text{base} : (n : I) \rightarrow \langle G\ n\rangle \rightarrow \oplus\text{HIT}\ I\ G$ 
   $\_+\oplus\_ : \oplus\text{HIT}\ I\ G \rightarrow \oplus\text{HIT}\ I\ G \rightarrow \oplus\text{HIT}\ I\ G$ 
   $+\oplus\text{Assoc} : \forall\ x\ y\ z \rightarrow x\ +\oplus\ (y\ +\oplus\ z) \equiv (x\ +\oplus\ y)\ +\oplus\ z$ 
   $+\oplus\text{Rid} : \forall\ x \rightarrow x\ +\oplus\ 0\oplus \equiv x$ 
   $+\oplus\text{Comm} : \forall\ x\ y \rightarrow x\ +\oplus\ y \equiv y\ +\oplus\ x$ 
   $\text{base}0\oplus : \forall\ n \rightarrow \text{base}\ n\ 0\oplus \langle G\ n\rangle \equiv 0\oplus$ 
   $\text{base}+\oplus :$ 
   $\forall\ n\ x\ y \rightarrow \text{base}\ n\ x\ +\oplus\ \text{base}\ n\ y \equiv \text{base}\ n\ (x\ +\oplus\ \langle G\ n\rangle\ y)$ 
   $\text{trunc} : \text{isSet}\ (\oplus\text{HIT}\ I\ G)$ 

```

In $\oplus\text{HIT}$, the direct sum is generated by homogeneous elements instead of being formed as a sum over elements in all degrees. The $0\oplus$ constructor is the neutral element of $\oplus\text{HIT}$, base forms a homogeneous element of degree n , and $_+\oplus_$ lets us take the sum of $\oplus\text{HIT}$ elements. The first three path constructors ensure that the resulting type is a commutative monoid, while the next two ensure that $0\oplus$ and $_+\oplus_$ reflect the 0 and $_+_$ of homogeneous elements. Finally, the last constructor is necessary in order to ensure that the type is a set. Without it, we could easily form non-trivial paths; for example, $+\oplus\text{Comm}\ x\ x$ would be a path from $x\ +\oplus\ x$ to itself which would not necessarily be equal to refl .

It is easy to give an abelian group structure to $\oplus\text{HIT}$ by defining an inverse by pattern-matching:

```

 $-\oplus\_ : \oplus\text{HIT}\ I\ G \rightarrow \oplus\text{HIT}\ I\ G$ 
 $-\oplus\ 0\oplus = 0\oplus$ 
 $-\oplus\ (\text{base}\ n\ x) = \text{base}\ n\ (-\langle G\ n\rangle\ x)$ 
 $-\oplus\ (x\ +\oplus\ y) = (-\oplus\ x)\ +\oplus\ (-\oplus\ y)$ 
 $-\oplus\ (+\oplus\text{Assoc}\ x\ y\ z\ i) = +\oplus\text{Assoc}\ (-\oplus\ x)\ (-\oplus\ y)\ (-\oplus\ z)\ i$ 
 $-\oplus\ (+\oplus\text{Rid}\ x\ i) = +\oplus\text{Rid}\ (-\oplus\ x)\ i$ 

```

```

 $-\oplus\ (+\oplus\text{Comm}\ x\ y\ i) = +\oplus\text{Comm}\ (-\oplus\ x)\ (-\oplus\ y)\ i$ 
 $-- \dots$ 

```

We omit the last three cases of the definition as they are a little bit more complicated. Essentially, the definition of $-\oplus$ proceeds directly by structural recursion, negating the homogeneous elements. It would be possible to add a constructor for $-\oplus$ to the HIT, however this would have forced us to handle even more cases when defining functions out of $\oplus\text{HIT}$. Moreover, as $-\oplus$ is a function we get many useful definitional equalities for concrete inputs.

Having defined negation, we can easily prove that $\oplus\text{HIT}$ is an abelian group for a general index type I . In order to avoid long definitions with many cases, we prove a special eliminator which only requires us to give the cases for point constructors when proving propositions. As $\oplus\text{HIT}$ is a set, all of the equations between $\oplus\text{HIT}$ elements are propositions, so this special eliminator shortens the proofs substantially and we can avoid a lot of boilerplate code. Defining special eliminators like this is a common pattern for working with set truncated HITs in the *agda/cubical* library.

Remark 3.1. We could have proved that $\oplus\text{HIT}\ \mathbb{N}$ and $\oplus\text{Fun}$ are equivalent types and that this equivalence preserves the raw group structure $(0, _+_, -_)$. This way, the SIP would have allowed us to get the proofs of the abelian group laws for $\oplus\text{HIT}\ \mathbb{N}$ for free. However, as the proofs of these laws are easy to do in the general case, using the special eliminator for proving propositions, we formalized it for $\oplus\text{HIT}\ I$ directly.

3.2 Graded rings

Having formalized direct sums, we now need to equip them with a ring structure induced by a graded multiplicative operation $\star : G_i \rightarrow G_j \rightarrow G_{i+j}$ over any monoid $(I, e, +)$. This means that we should define a product operation of type

$$\bigoplus_{i:I} G_i \rightarrow \bigoplus_{i:I} G_i \rightarrow \bigoplus_{i:I} G_i$$

using \star . For this product to satisfy the ring laws, the \star operation has to satisfy suitable graded versions of them, e.g. it has to have a multiplicative unit in degree 0, be associative, distribute over sums of elements of the same degree, etc. We refer the interested reader to the formalization for the exact formulation of these laws.

The \star operation over the monoid $(\mathbb{N}, 0, +)$ can be lifted pointwise to a product operation on $\oplus\text{Fun}$ by the following definition:

$$(f * g)(n) = \sum_{i=0}^n \uparrow_i^n (f(i) \star g(n-i))$$

Here \uparrow_i^n is a transport from $G_{i+(n-i)}$ to G_n . This transport is necessary for Agda to see that this is a well-defined sum of elements in G_n . It is straightforward to show that if f and g are zero after k respectively k' steps, then the product is

zero after at most $k \cdot k'$ steps. So $f * g$ is indeed a well-defined element of $\oplus \text{Fun}$.

Using the graded ring laws for \star one can then prove the ring laws for $*$. However, doing this gets surprisingly complicated already for associativity as we need to fill in the gaps in the following chain of equalities:

$$\begin{aligned}
& (f * (g * h))(n) \\
&= \sum_{i=0}^n \uparrow_i^n (f(i) \star (g * h)(n-i)) \\
&= \sum_{i=0}^n \uparrow_i^n \left(f(i) \star \left(\sum_{j=0}^{n-i} \uparrow_j^{n-i} (g(j) \star h(n-i-j)) \right) \right) \\
&= \dots \\
&= \sum_{i=0}^n \uparrow_i^n \left(\left(\sum_{j=0}^i \uparrow_j^i (f(j) \star g(i-j)) \right) \star h(n-i) \right) \\
&= \sum_{i=0}^n \uparrow_i^n ((f * g)(i) \star h(n-i)) \\
&= ((f * g) * h)(n)
\end{aligned}$$

This might not look too bad on paper, especially if one ignores the transports, but when one starts formalizing it in Cubical Agda, one quickly finds oneself descending deeper and deeper into transport hell. This is exactly where $\oplus \text{HIT}$ shines; by instead working mainly with homogeneous elements, we can lift \star (for any monoid I) and avoid all transports in the definition by writing the following:⁴

```

_★_ :  $\oplus \text{HIT } I \ G \rightarrow \oplus \text{HIT } I \ G \rightarrow \oplus \text{HIT } I \ G$ 
0 $\oplus$  * _ = 0 $\oplus$ 
(base n x) * 0 $\oplus$  = 0 $\oplus$ 
(base n x) * (base m y) = base (n · m) (x ★ y)
(base n x) * (y + $\oplus$  z) = (base n x * y) + $\oplus$  (base n x * z)
(x + $\oplus$  y) * z = (x * z) + $\oplus$  (y * z)
-- cases for higher constructors omitted

```

The most interesting case of the definition is the one where both arguments are `base` elements. This corresponds to taking the product of homogeneous elements, which of course gives another homogeneous element, defined using \star .

This gives a construction of the product operation, but we still have to prove that it satisfies the ring laws. These laws are propositions as they are equations between elements of a set, so we can again use the special eliminator for proving proposition. It hence suffices to prove the laws for the point constructors: `0 \oplus` , `base`, and `+ \oplus` . Generally, the `0 \oplus` and `+ \oplus` cases are trivial, which leaves the cases for `base`. This means that we are left to prove the laws for homogeneous elements, which directly follows from the graded ring laws for \star . In particular, associativity follows directly.

⁴In practice, we give the same definition using pre-defined recursors for $\oplus \text{HIT}$ which helps convince Agda's termination checker. The definition by pattern-matching is merely included for pedagogical reasons.

Remark 3.2. The Lean Mathlib [29] also has a formalization of direct sums and graded rings. There, $\bigoplus_{i \in I} G_i$ is defined as finitely supported dependent functions, i.e. dependent functions that are nonzero only at a finite set of points, just like in the first definition above. In fact, there is even a formal proof that a definition of $*$, similar to the sum involving transports, is associative. This proof heavily relies on tactics and automation to handle the complexity induced by the transports. However, Cubical Agda has very limited support for automation and because Cubical Agda is proof relevant, this approach would most likely be a lot more challenging than in the proof irrelevant setting of Lean.

3.2.1 Transporting the ring structure using the SIP.

Having established that $\oplus \text{HIT}$ can be turned into a ring given \star , it is natural to ask how this relates to the ring structure on $\oplus \text{Fun}$ using $*$. If we were to go through the hurdles of transport hell, we would be able to equip $\oplus \text{Fun}$ with a ring structure and, with some more work, we can then prove that this is isomorphic to $\oplus \text{HIT } \mathbb{N}$. The SIP then implies that these rings are path-equal which means that all properties proved for one of them also holds for the other. However, as discussed in Section 2.2, the SIP actually gives us even more. If we, instead of proving the ring laws for $*$ by hand, would prove that $\oplus \text{HIT } \mathbb{N}$ and $\oplus \text{Fun}$ are equivalent as types and that this equivalence preserves the *raw* ring structure $(0, 1, +, *)$, then we could use the SIP to transport also the ring laws to get an induced ring structure on $\oplus \text{Fun}$.

Proving that $\oplus \text{HIT } \mathbb{N}$ and $\oplus \text{Fun}$ are equivalent types can be done quite easily. In one direction, we send `0 \oplus` and `+ \oplus` to the zero and addition of $\oplus \text{Fun}$, and `base n x` to the function that is x at position n and zero otherwise. For the other direction, we can map a function to a nested `+ \oplus` sum of `base` elements. It is direct to prove that these maps cancel and hence establish that the types are equivalent. We can now show that the raw ring structures is preserved by the equivalence. This is straightforward for all cases except multiplication, which as usual boils down to homogeneous elements. The proofs are a bit repetitive, but do not present any special technical or conceptual difficulties. The SIP then gives an induced ring structure on $\oplus \text{Fun}$ which is path-equal to the one on $\oplus \text{HIT } \mathbb{N}$, and we hence get that the rings satisfy the same properties. This can be seen as a practical example, in the spirit of CoqEAL [11], of using the SIP to avoid long and technical proofs by changing data representation to a type where some proofs are easier.

4 Polynomial rings

Recall that $R[X]$ can be defined by letting the underlying abelian group be $\bigoplus_{\mathbb{N}} R$ and the multiplication be induced by the one of R . Having formalized two constructions of $\bigoplus_{\mathbb{N}}$, we hence get two equivalent representations of $R[X]$ which we will now compare in detail. We will also compare them with a list based representation, which clarifies the

role of decidable equality in the formalization, and discuss how $R[X_1, \dots, X_n]$ can be directly encoded using $\oplus\text{HIT}$.

4.1 Polynomials as graded rings

As discussed in the previous section, we can easily transport properties back and forth between the $\oplus\text{Fun}$ and $\oplus\text{HIT}$ representations of $R[X]$. In practice, however, the two representations are quite different to work with. As we have seen, the one based on $\oplus\text{Fun}$ is especially well-suited for operations that can be defined pointwise, e.g. addition, but for operations that are not pointwise, things can get quite involved, because they may depend on the k after which the function becomes 0. For example, in order to evaluate a polynomial at a point, one needs to take a sum up to k ; as we only merely have access to k , this gets somewhat complicated. The $\oplus\text{HIT}$ definition has the benefit that it is inductive, which means that operations can be defined by recursion and properties proved by structural induction. This means that some definitions, like addition, get a bit longer than for $\oplus\text{Fun}$, as we need to write more cases. On the other hand, polynomial evaluation is much easier to implement.

Also, from a computational perspective, the two representations behave differently. The $\oplus\text{Fun}$ definition can be seen as a shallow embedding, since polynomials are encoded as Agda functions, which makes them harder both to input and print. It is also complicated to predict how efficient they are to compute with and how much memory they use, as this depends on the internal representation of Agda functions.

The HIT definition, on the other hand, is a deep embedding, which is easy to write and print. Furthermore, it is a sparse representation where polynomials like $2X^3 + X^{100}$ are generated by monomials and can be written compactly as

```
base 3 2 +⊕ base 100 1
```

This resembles the polynomials on sparse Horner normal form of Grégoire and Mahboubi [17, Section 3]. In this representation, a polynomial is either a constant $a \in R$ or on the form $a + pX^n$ for $a \in R$ and p a polynomial in sparse Horner normal form. The above polynomial would hence be encoded as $0 + (2 + X^{97})X^3$, making it similar in terms of memory usage to $\oplus\text{HIT } \mathbb{N}$. It would be possible to implement this representation as a HIT where equal representations of the same polynomial get identified, and then prove that it is equivalent to $\oplus\text{HIT } \mathbb{N}$. However, a drawback of these sparse representations is that it is easy to get complicated terms when working with concrete polynomials. For example,

```
base 3 2 +⊕ base 3 3
```

represents the same polynomial as `base 3 5` up to a path. To remedy this, one should be careful to normalize polynomials appropriately when doing efficient computations [17, Section 3.2], which in turn might require some decidability assumptions on R (e.g. to simplify `base n 0r` to `0⊕`).

4.2 List based polynomials and decidable equality

Another common representation of polynomials is as a list of coefficients; for instance, `[1, 0, 2, 5]` represents $1 + 2X^2 + 5X^3$. However, one has to be careful to avoid zeroes at the end of the list, as for example `[1, 0, 2, 5, 0, 0]` encodes the same polynomial as the list above. This can be easily fixed by quotienting by a relation which equates lists modulo trailing zeroes. The following HIT encodes this very compactly:

```
data ListPoly : Type where
  [] : ListPoly
  _::_ : (x : R) → (xs : ListPoly) → ListPoly
  drop0 : 0r :: [] ≡ []
```

Proving $xs \equiv xs ++ \text{replicate } n \text{ } 0r$ for $xs : \text{ListPoly}$ and $n : \mathbb{N}$ is easy, so this indeed equates a list with itself appended by arbitrarily many zeroes. Furthermore, by a rather ingenious proof, one can show that `ListPoly` is equivalent to $\oplus\text{Fun}$, which justifies omitting the set truncation in the definition; thus, this is an alternative definition of $R[X]$. However, it is not well-suited for sparse polynomials, as e.g. $2X^3 + X^{100}$ is encoded by a list with 101 elements, out of which 99 are 0. Nevertheless, for dense polynomials it is not so bad and we avoid having to normalize monomials for efficiency.

Something remarkable with all three of these representations is that we avoid the assumption that R is discrete (i.e. that its underlying type has decidable equality), while still obtaining constructive proofs that $R[X]$ is a commutative ring. This crucially relies on HITs for constructing quotient types. However, as is common in constructive algebra [25], we cannot compute neither the degree nor the leading coefficient of these HIT polynomials. This is especially clear for `ListPoly`, where we cannot know how many trailing zeroes a list has without being able to test if an element is zero. However, this is not as big a problem as one might expect. Indeed, as both the `ListPoly` and $\oplus\text{HIT}$ are HITs, we can construct functions on them by structural recursion and reason about them by induction without having to talk about degrees.

For some applications it might be necessary to compute the degree or leading coefficient though (e.g. for polynomial pseudo division [19]). To this end, it is useful to restrict attention to discrete rings R and define $R[X]$ as lists with a nonzero last coefficient, which makes the degree and leading coefficient computable. This is the approach of the `MathComp` library in `Coq` which was used by Gonthier et al. [16] to formalize the Feit-Thompson odd order theorem and many other impressive results in algebra. This representation can be seen as a normal form for the other representations considered here. Indeed, when R is discrete, it is easy to reduce `ListPoly` to this form by dropping trailing zeroes. By the equivalence of `ListPoly` with both $\oplus\text{HIT } \mathbb{N}$ and $\oplus\text{Fun}$, these can also be normalized to this form.

However, a caveat with representing polynomials on this normalized form is that the invariant of having a nonzero

last element has to be preserved by all operations, complicating for example addition, since trailing zeroes might have to be removed after adding the lists. One also has to be careful not to have Agda normalize the proofs that the last element is nonzero. This also applies to $\oplus\text{Fun}$, which is also represented as a subtype. However, it does not apply to $\oplus\text{HIT}$ and ListPoly , as these are HITs where we instead work directly with representatives of equivalence classes of polynomials.

4.3 Multivariate polynomials

Having discussed these representations of univariate polynomials, it is natural to also consider multivariate polynomial rings $R[X_1, \dots, X_n]$. These will be used in Section 5 where we will prove that some cohomology rings are equivalent to multivariate polynomial rings quotiented by ideals.

Often in algebra, $R[X_1, \dots, X_n]$ is represented as iterated univariate polynomials $((R[X_1])[X_2]) \cdots [X_n]$. This has the benefit that properties can be proved for $R[X_1, \dots, X_n]$ by proving that if the property holds for R , then it also holds for $R[X]$, and then applying this n times. While this is convenient for proving properties of $R[X_1, \dots, X_n]$, it is not always the best for doing computations.

To remedy this, we can let the indexing set be $\text{Vec } n \mathbb{N}$ in $\oplus\text{HIT}$ and obtain a direct definition of $R[X_1, \dots, X_n]$. This definition still gives a simple representation of polynomials where the base constructor represents monomials; for instance, $\text{base } (4, 0, 3) a$ corresponds to $aX_1^4X_3^3 \in R[X_1, X_2, X_3]$. Having a direct definition of $R[X_1, \dots, X_n]$ is good for computations as operations like multiplication become simpler if they are directly defined instead of obtained by iterating the algorithm for univariate polynomials. It also helps with proofs, e.g. proving formally that

$$R[X_1, \dots, X_n]/(X_1, \dots, X_n) \cong R$$

using iterated univariate polynomials would be very cumbersome, but with the Vec based representation it is easily proved by $\oplus\text{HIT}$ induction.

For this reason, we are going to use $\oplus\text{HIT } (\text{Vec } n \mathbb{N})$ as our representation of $R[X_1, \dots, X_n]$ in the next section. To avoid confusion when working with two notions defined using $\oplus\text{HIT}$ (i.e. polynomials and cohomology rings), we will from now on simply write e.g. aX^2Y^3 for $\text{base } (2, 3) a$ and use 0 for $0\oplus$ as well as $+$ for $+\oplus$ when working with $R[X_1, \dots, X_n]$.

5 Cohomology rings

We are now finally ready to formalize cohomology rings. Recall, we define the cohomology ring of X with coefficients in a ring R by $H^*(X, R) = \bigoplus_{i \in \mathbb{N}} H^i(X, R)$. In the formalization, we use $\oplus\text{HIT}$ to formalize $H^*(X, R)$; the fact that it is a ring follows immediately by construction, using the graded ring laws of \smile_h .

Note that it is crucial that our definition of \bigoplus avoided assumptions about decidable equality — it is not the case that $H^i(X, R)$ has decidable equality for any i, X and R . For

a simple counterexample, pick a ring R without decidable equality (e.g. $R = \mathbb{R}$) and consider the cohomology group $H^n(\mathbb{S}^n, R)$. As it is isomorphic (as an abelian group) to R , it cannot have decidable equality, thereby preventing us from even defining $H^*(\mathbb{S}^n, R)$.

We will now discuss the various formalized computations of $H^*(X, R)$ for different spaces X and rings R which we have carried out. We have formalized these computations as equivalences of *rings*, but we make explicit in the proofs how they respect the graded structure. Polynomial rings are particularly useful in the computation of these cohomology rings. While not all cohomology rings are isomorphic to $R[X_1, \dots, X_n]/I$ for some commutative ring R and ideal I , there are many examples where a description using polynomial rings is arguably the most tractable choice. These various computations will allow us to conclude that various spaces are not equivalent, despite having the same cohomology groups.

5.1 Spheres

As a warm-up, let us consider the integral cohomology ring of \mathbb{S}^1 .

Proposition 5.1. *We have $H^*(\mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z}[X]/(X^2)$.*

Proof. The cohomology groups of \mathbb{S}^1 are easily proved to be:

$$H^n(\mathbb{S}^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

For a synthetic proof of this, see e.g. Brunerie et al. [5, Section 5]. For $i \leq 1$, let $\phi_i : \mathbb{Z} \cong H^i(\mathbb{S}^1, \mathbb{Z})$ be the above isomorphisms. We define a map $\psi : \mathbb{Z}[X] \rightarrow H^*(\mathbb{S}^1, \mathbb{Z})$ by:

$$\begin{aligned} \psi(0) &= 0\oplus \\ \psi(cX^0) &= \text{base } 0 (\phi_0(c)) \\ \psi(cX^1) &= \text{base } 1 (\phi_1(c)) \\ \psi(cX^{2+n}) &= 0\oplus \\ \psi(p + q) &= \psi(p) + \psi(q) \\ &\vdots \end{aligned}$$

The remaining cases are immediate by construction as we are defining a map from $\oplus\text{HIT}$ to itself, and since ϕ_i is a family of group homomorphisms. Proving that this map is a ring homomorphism is easy, since one only has to consider the cup product in low dimensions, in which case it simply corresponds to left- or right-multiplication over \mathbb{Z} . Moreover, as ψ cancels on X^2 , this allows us to view ψ as a map $\mathbb{Z}[X]/(X^2) \rightarrow H^*(\mathbb{S}^1, \mathbb{Z})$. Its inverse $\psi^{-1} : H^*(\mathbb{S}^1, \mathbb{Z}) \rightarrow$

$\mathbb{Z}[X]/(X^2)$ is given (on homogeneous elements) by:

$$\begin{aligned}\psi^{-1}(0\oplus) &= 0 \\ \psi^{-1}(\mathbf{base}\ 0\ c) &= (\phi_0^{-1}(c))X^0 \\ \psi^{-1}(\mathbf{base}\ 1\ c) &= (\phi_1^{-1}(c))X^1 \\ \psi^{-1}(\mathbf{base}\ (2+n)\ c) &= 0 \\ \psi^{-1}(x+\oplus\ y) &= \psi^{-1}(x) + \psi^{-1}(y)\end{aligned}$$

That ψ and ψ^{-1} cancel follows by construction. \square

The above proof also works for higher spheres. We define these by $(n-1)$ -fold suspension of \mathbb{S}^1 , where the suspension of a type is defined in Cubical Agda by

```
data Susp (A : Type) : Type where
  north : Susp A
  south : Susp A
  merid : A → north ≡ south
```

Let us explicitly define three recurring special cases:

$$\mathbb{S}^2 = \Sigma \mathbb{S}^1 \quad \mathbb{S}^3 = \Sigma \mathbb{S}^2 \quad \mathbb{S}^4 = \Sigma \mathbb{S}^3$$

As our cohomology theory satisfies the Eilenberg-Steenrod axioms⁵, it follows from the suspension axiom that, for $n \geq 1$, we have

$$H^m(\mathbb{S}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & m = 0, n = m \\ 0 & \text{otherwise} \end{cases}$$

Hence, by a similar proof to that of Proposition 5.1, we may conclude the following:

Proposition 5.2. *For $n \geq 1$, we have $H^*(\mathbb{S}^n, \mathbb{Z}) \cong \mathbb{Z}[X]/(X^2)$.*

We remark that the exact same method may be used to compute the cohomology rings of spheres with arbitrary coefficients, although this has not yet been formalized.

5.2 The complex projective plane

For a slightly more technical example, let us consider $\mathbb{C}P^2$. We define it in terms of its usual CW decomposition (see e.g. Hatcher [18, Example 0.6]), using the following homotopy pushout:

$$\begin{array}{ccc} \mathbb{S}^3 & \longrightarrow & \{*\} \\ \text{hopf} \downarrow & & \downarrow \\ \mathbb{S}^2 & \longrightarrow & \mathbb{C}P^2 \end{array}$$

Above, $\text{hopf} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is the generator of $\pi_3(\mathbb{S}^2)$. In Cubical Agda, we may define $\mathbb{C}P^2$ as the following HIT:

```
data CP2 : Type where
  cfbase : CP2
  cfcod : S2 → CP2
  cfglue : (x : S3) → cfcod (hopf x) ≡ cfbase
```

⁵See e.g. Cavallo [9] for the definition. The fact that our theory satisfies the axioms was proved by Brunerie et al. [5].

For the construction of `hopf`, see Brunerie [4, Chapter 2.5]. The cup product on $\mathbb{C}P^2$ was shown by Brunerie [4, Proposition 6.3.2], for the first time in HoTT/UF, to induce the usual multiplication on \mathbb{Z} via

$$\mathbb{Z} \times \mathbb{Z} \cong H^2(\mathbb{C}P^2) \times H^2(\mathbb{C}P^2) \xrightarrow{\smile_h} H^4(\mathbb{C}P^2) \cong \mathbb{Z}$$

This was later also formalized by Brunerie et al. [5]. The ring structure on $H^*(\mathbb{C}P^2, \mathbb{Z})$ is particularly interesting, since it plays a fundamental role in Brunerie's synthetic proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}_2$. We can now package this up into a more compact statement.

Proposition 5.3. $H^*(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}[X]/(X^3)$

Proof. The groups $H^n(\mathbb{C}P^2, \mathbb{Z})$ are described in Table 2.

Dimension	Isomorphic to	Generator
H^0	\mathbb{Z}	η
H^2	\mathbb{Z}	α
H^4	\mathbb{Z}	β
$H^n, n \neq 0, 2, 4$	0	0

Table 2. Cohomology groups of $\mathbb{C}P^2$.

Let us denote by ϕ_i the isomorphisms $\mathbb{Z} \cong H^i(\mathbb{C}P^2, \mathbb{Z})$, for $i \in \{0, 2, 4\}$. We define $\psi : \mathbb{Z}[X] \rightarrow H^*(\mathbb{C}P^2, \mathbb{Z})$ by:

$$\begin{aligned}\psi(0) &= 0\oplus \\ \psi(cX^0) &= \mathbf{base}\ 0\ (\phi_0(c)) \\ \psi(cX^1) &= \mathbf{base}\ 2\ (\phi_2(c)) \\ \psi(cX^2) &= \mathbf{base}\ 4\ (\phi_4(c)) \\ \psi(cX^{3+n}) &= 0\oplus \\ \psi(p+q) &= \psi(p) +\oplus\ \psi(q) \\ &\vdots\end{aligned}$$

The left out cases are automatic since the above, by definition, is a homomorphism of abelian groups. Proving that the map is a ring homomorphism boils down to showing one non-trivial identity:

$$\psi(c_1X^1 \cdot c_2X^1) \equiv \psi(c_1X^1) \smile_h \psi(c_2X^1)$$

This follows by $\alpha \smile_h \alpha \equiv \beta$, which can be shown by an application of the Gysin sequence (see [4, Chapter 6]). We have:

$$\begin{aligned}\psi(c_1X^1 \cdot c_2X^1) &\equiv \mathbf{base}\ 4\ (\phi_4(c_1c_2)) \\ &\equiv c_1c_2 \cdot \mathbf{base}\ 4\ (\phi_4(1)) \\ &\equiv c_1c_2 \cdot \mathbf{base}\ 4\ \beta \\ &\equiv c_1c_2 \cdot \mathbf{base}\ 4\ (\alpha \smile_h \alpha) \\ &\equiv c_1c_2 \cdot (\mathbf{base}\ 2\ \alpha \smile_h \mathbf{base}\ 2\ \alpha) \\ &\equiv c_1c_2 \cdot (\mathbf{base}\ 2\ (\phi_2\ 1) \smile_h \mathbf{base}\ 2\ (\phi_2\ 1)) \\ &\equiv \mathbf{base}\ 2\ (\phi_2(c_1)) \smile_h \mathbf{base}\ 2\ (\phi_2(c_2)) \\ &\equiv \psi(c_1X^1) \smile_h \psi(c_2X^1)\end{aligned}$$

Moreover, as $\psi(X^3) = 0 \oplus$, we may view ψ as a homomorphism $\mathbb{Z}[X]/(X^3) \rightarrow H^*(\mathbb{C}P^2, \mathbb{Z})$. For its inverse, one proceeds just like in Proposition 5.1, and proving that the maps cancel is equally straightforward. \square

Cohomology rings are particularly useful for distinguishing “similar” spaces. Compare, for instance, $\mathbb{C}P^2$ with the wedge sum $\mathbb{S}^2 \vee \mathbb{S}^4$, the space obtained by gluing together the base points of \mathbb{S}^2 and \mathbb{S}^4 . In Cubical Agda, we may define the wedge sum of two pointed types by

```
data _v_ (A B : Pointed) : Type where
  inl : typ A → A ∨ B
  inr : typ B → A ∨ B
  push : inl (pt A) ≡ inr (pt B)
```

The cohomology groups of $\mathbb{C}P^2$ and $\mathbb{S}^2 \vee \mathbb{S}^4$ are the same in each dimension. There is an intuitive reason for this: $\mathbb{S}^2 \vee \mathbb{S}^4$ has one connected component, one 3-dimensional hole, one 5-dimensional hole, and nothing more. Hence, the cohomology groups should coincide with those of $\mathbb{C}P^2$. However, the two spaces are not homotopy equivalent. One way to see this is to define a predicate $P : \mathbb{N}^2 \times \text{Type} \rightarrow \mathbf{hProp}$ such that $P(n, m, X)$ expresses that the cup product $\smile_h : H^n(X, \mathbb{Z}) \times H^m(X, \mathbb{Z}) \rightarrow H^{n+m}(X, \mathbb{Z})$ is trivial. For our two spaces, we can show that $P(2, 2, \mathbb{S}^2 \vee \mathbb{S}^4)$ holds, while $P(2, 2, \mathbb{C}P^2)$ does not. This method was used by Brunerie et al. [5] to distinguish the torus from $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$. This approach is, however, rather ad hoc, as the predicate P will have to be adjusted if we want to capture other aspects of \smile_h than triviality. With cohomology rings, we may instead simply conclude that $\mathbb{C}P^2 \neq \mathbb{S}^2 \vee \mathbb{S}^4$ from the fact that $H^*(\mathbb{C}P^2, \mathbb{Z}) \neq H^*(\mathbb{S}^2 \vee \mathbb{S}^4, \mathbb{Z})$.

Proposition 5.4. $H^*(\mathbb{S}^2 \vee \mathbb{S}^4, \mathbb{Z}) \cong \mathbb{Z}[X, Y]/(X^2, XY, Y^2)$

Proof. Let ϕ_i again be the isomorphisms $\mathbb{Z} \cong H^n(\mathbb{S}^2 \vee \mathbb{S}^4, \mathbb{Z})$, for $i \in \{0, 2, 4\}$. We define $\psi : \mathbb{Z}[X, Y] \rightarrow H^*(\mathbb{S}^2 \vee \mathbb{S}^4, \mathbb{Z})$ by

$$\begin{aligned} \psi(0) &= 0 \oplus \\ \psi(c) &= \mathbf{base\ 0}(\phi_0(c)) \\ \psi(cX) &= \mathbf{base\ 2}(\phi_2(c)) \\ \psi(cY) &= \mathbf{base\ 4}(\phi_4(c)) \\ \psi(cX^n Y^m) &= 0 \oplus && \text{if } n + m \geq 2 \\ \psi(p + q) &= \psi(p) + \oplus \psi(q) \end{aligned}$$

That ψ induces an isomorphism $\mathbb{Z}[X, Y]/(X^2, XY, Y^2) \cong H^*(\mathbb{S}^2 \vee \mathbb{S}^4, \mathbb{Z})$ is equally straightforward to verify as in the proofs of Propositions 5.1 and 5.3. \square

Combining Propositions 5.3 and 5.4 we get the following:

Corollary 5.5. $\mathbb{C}P^2 \neq \mathbb{S}^2 \vee \mathbb{S}^4$

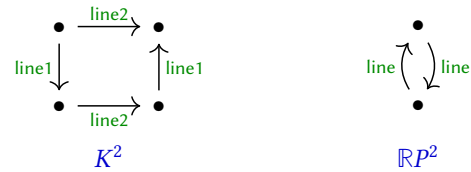
5.3 The Klein bottle and the real projective plane with an adjoined circle

Sometimes integral cohomology rings are not enough to distinguish spaces and we have to change the coefficient ring to something else than \mathbb{Z} . For an example of this, let us introduce two new spaces: the Klein bottle K^2 and the real projective plane $\mathbb{R}P^2$. In Cubical Agda, they are defined by:

```
data K2 : Type where
  point : K2
  line1 : point ≡ point
  line2 : point ≡ point
  square : PathP (λ i → line1 (~ i) ≡ line1 i) line2 line2
```

```
data RP2 : Type where
  point : RP2
  line : point ≡ point
  square : line ≡ sym line
```

The intuition behind these HITs is that they precisely capture the usual folding diagrams representing each space.



Consider K^2 and $\mathbb{R}P^2 \vee \mathbb{S}^1$. The two spaces both have integral cohomology groups

$$H^n(K^2, \mathbb{Z}) \cong H^n(\mathbb{R}P^2 \vee \mathbb{S}^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n \leq 1 \\ \mathbb{Z}_2 & n = 2 \\ 0 & n > 2 \end{cases}$$

One can show that $\smile_h : H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ vanishes for both spaces. Thus, we can easily show the following:

Proposition 5.6. *We have the following isomorphisms:*

$$H^*(K^2, \mathbb{Z}) \cong H^*(\mathbb{R}P^2 \vee \mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z}[X, Y]/(X^2, XY, 2Y, Y^2)$$

Proof. Let S be either of the two spaces in question. The proof is close to identical to that of Proposition 5.3 – we let X correspond to the generator of $H^1(S, \mathbb{Z})$ and Y correspond to the generator of $H^2(S, \mathbb{Z})$. The only difference is the 2-torsion in $H^2(S, \mathbb{Z})$ for both spaces, which is accounted for by the factor of $2Y$ in the quotient. \square

We can, however, distinguish K^2 and $\mathbb{R}P^2 \vee \mathbb{S}^1$ by computing their cohomology rings with \mathbb{Z}_2 coefficients. In this case, both spaces still have the same non-trivial cohomology groups, as described in Table 3.

The characterization of the cup product on the rings is non-trivial and will have to be covered in a separate paper.

Dim.	Space	Isomorphic to	Generators
H^0	K^2 $\mathbb{R}P^2 \vee S^1$	\mathbb{Z}_2	η_1 η_2
H^1	K^2 $\mathbb{R}P^2 \vee S^1$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	α_1, β_1 α_2, β_2
H^2	K^2 $\mathbb{R}P^2 \vee S^1$	\mathbb{Z}_2	γ_1 γ_2

Table 3. Cohomology groups of K^2 and $\mathbb{R}P^2 \vee S^1$.

For K^2 it is described, on generators, by

$$\alpha_1^2 \equiv \alpha_1 \smile_h \beta_1 \equiv \beta_1 \smile_h \alpha_1 \equiv \gamma_1 \quad \text{and} \quad \beta_1^2 \equiv 0. \quad (1)$$

On $\mathbb{R}P^2 \vee S^1$ it is described by

$$\alpha_2 \smile_h \beta_2 \equiv \beta_2 \smile_h \alpha_2 \equiv \beta_2^2 \equiv 0 \quad \text{and} \quad \alpha_2^2 \equiv \gamma_2. \quad (2)$$

These relations let us describe the two cohomology rings.

Proposition 5.7. $H^*(K^2, \mathbb{Z}_2) \cong \mathbb{Z}_2[X, Y]/(X^3, Y^2, X^2 + XY)$

Proof. In order to define $\psi : \mathbb{Z}_2[X, Y] \rightarrow H^*(K^2, \mathbb{Z}_2)$, let us, for the sake of conciseness, be somewhat more informal than before. It suffices to define it on generators, i.e. 1, X and Y :

$$\begin{aligned} \psi(1) &= \text{base } 0 \eta_1 \\ \psi(X) &= \text{base } 1 \alpha_1 \\ \psi(Y) &= \text{base } 1 \beta_1 \end{aligned}$$

This induces the rest of the definition as ψ needs to be a group homomorphism. Moreover, by the previous equations ψ is automatically a ring homomorphism. We observe that ψ vanishes on $(X^3, Y^2, X^2 + XY)$:

$$\begin{aligned} \psi(X^3) &= \text{base } 3 (\alpha_1^3) = \text{base } 3 0_h = 0 \oplus \\ \psi(Y^2) &= \text{base } 3 (\beta_1^2) = \text{base } 2 0_h = 0 \oplus \\ \psi(X^2 + XY) &= \psi(X^2) + \psi(XY) = \text{base } 2 (\alpha_1^2 +_h (\alpha_1 \smile_h \beta_1)) \\ &= \text{base } 2 (\alpha_1^2 +_h \alpha_1^2) = \text{base } 2 0_h \\ &= 0 \oplus \end{aligned}$$

The first of these identities follows from $H^3(K^2, \mathbb{Z}_2)$ being trivial, while the other two follow from the relations in (1) and the fact that the coefficients are \mathbb{Z}_2 .

Hence, we may view ψ as a homomorphism

$$\mathbb{Z}_2[X, Y]/(X^3, Y^2, X^2 + XY) \rightarrow H^*(K^2, \mathbb{Z}_2)$$

The fact that it is an isomorphism is a straightforward consequence of the identities described in (1). \square

Proposition 5.8. *We have the following isomorphism:*

$$H^*(\mathbb{R}P^2 \vee S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2[X, Y]/(X^3, XY, Y^2)$$

Proof. Similar to the proof of Proposition 5.7, using the identities described in (2). \square

And thus, since the two spaces in question have different cohomology rings, we arrive at the main result of this section.

Corollary 5.9. $K^2 \neq \mathbb{R}P^2 \vee S^1$

6 Conclusions and future work

In this paper, we have developed cohomology rings synthetically and made multiple computations of such rings. To the best of our knowledge, no other systems has a formalization of cohomology rings for some cohomology theory. We have relied heavily on features of HoTT/UF, including univalence and HITs, to do this in a way that is fully constructive without sacrificing generality. The formalization also relies on the cubical SIP of Angiuli et al. [1] to conveniently transport proofs between equivalent structured types when developing the theory about direct sums in order to get the best out of both representations. The fact that this was done in Cubical Agda where the univalence axiom, and hence the SIP, has computational content means that it can be seen as a form of *data refinement* where we change data representation to facilitate easier proofs while not sacrificing computation.

This form of data refinement is similar to the one in CoqEAL [11] and in the Isabelle/HOL tool Autoref [20]. One difference in our work is that we not only refine programs, but also proof terms. Furthermore, both $\oplus\text{Fun}$ and $\oplus\text{HIT}$ are *proof-oriented* types in that they are both well-suited for doing different kinds of proofs, while in CoqEAL the refinement was done from proof-oriented to computation-oriented types. Another interesting difference is that HITs can be both proof-oriented and computation-oriented at the same time. For example, it is not necessary to refine `ListPoly` to ordinary lists as concrete computations will only involve the point constructors. Exploring these observations further is planned for future work.

In Table 4, we have summarized the computations of cohomology rings which we have formalized. The additions to the library required for these computation amount to about 8800 lines of code. However, this number should be taken with a grain of salt, as this project was not a standalone development. In addition to these 8800 lines of code, various already existing parts of the library had to be extended and rewritten, so the actual number of lines written is bigger.

At the moment, our computations of cohomology rings are somewhat ad-hoc, as each ring isomorphism is given by directly defining a homomorphism with an explicitly constructed inverse. It seems plausible that some of the complexity could be handled more conveniently by instead first defining $f : R[X_1, \dots, X_n] \rightarrow H^*(X, R)$ and then establishing that it is surjective. This would mean that $H^*(X, R) \cong R[X_1, \dots, X_n]/\ker(f)$ and the problem would be reduced to computing the generators of $\ker(f)$. Classically, one would typically rely on R being Noetherian, which carries over to

Space	Coefficients	Cohomology ring
S^n	\mathbb{Z}	$\mathbb{Z}[X]/(X^2)$
CP^2	\mathbb{Z}	$\mathbb{Z}[X]/(X^3)$
$S^2 \vee S^4$	\mathbb{Z}	$\mathbb{Z}[X, Y]/(X^2, XY, Y^2)$
K^2	\mathbb{Z}	$\mathbb{Z}[X, Y]/(X^2, XY, 2Y, Y^2)$
	\mathbb{Z}_2	$\mathbb{Z}_2[X, Y]/(X^3, Y^2, X^2 + XY)$
$RP^2 \vee S^1$	\mathbb{Z}	$\mathbb{Z}[X, Y]/(X^2, XY, 2Y, Y^2)$
	\mathbb{Z}_2	$\mathbb{Z}_2[X, Y]/(X^3, Y^2, XY)$

Table 4. Summary of cohomology rings

$R[X_1, \dots, X_n]$ by Hilbert’s basis theorem. This then implies that the kernel is finitely generated. This does, however, not work very well constructively and we would not be able to extract a list of generators of the ideal from the proof. Instead, we would have to rely on the theory of coherent rings which has previously been formalized in Coq by Coquand et al. [13], but in order to use it in this context we would need a proof that $R[X_1, \dots, X_n]$ is coherent, which boils down to Gröbner basis computations [22]. Such an approach, while interesting, would be a very major undertaking.

A less challenging direction of future work would be to compute cohomology rings by hand for some more spaces. For example, the cohomology ring of the torus should be very feasible to compute using our machinery. A much more ambitious project would then be to generalize this to a Künneth formula for computing cohomology of general products of spaces. Another class of spaces to consider is that of higher-dimensional real/complex projective spaces. Buchholtz and Rijke [7] gave a HoTT formalization of RP^n and RP^∞ , so it would be interesting to formalize also these spaces and compute their cohomology groups and rings.

In [5] reduced cohomology groups were also formalized. These too could be lifted to construct reduced cohomology rings $\tilde{H}^*(X, R)$. Such a formalization would probably benefit from the SIP as we could transport operations and results from $H^*(X, R)$ in order to obtain a compact construction of $\tilde{H}^*(X, R)$. This could then be used to give general theorems for computing the reduced cohomology ring of wedge sums:

$$\tilde{H}^*(X \vee Y, R) \cong \tilde{H}^*(X, R) \times \tilde{H}^*(Y, R)$$

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