Computing Cohomology Rings in Cubical Agda

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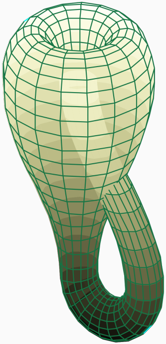
1. Cohomology and Cohomology Rings ?

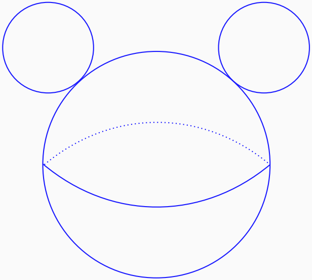
Klein Bottle vs Mickey Mouse

Question :

- How to prove that two topological spaces are not isomorphic ?



Klein Bottle $\mathbb{K}^2$



"Mickey Mouse space"

$\mathbb{S}^2 \bigvee \mathbb{S}^1 \bigvee \mathbb{S}^1$

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- To each topological space X, associate a sequence of abelian group $(H^i(X))_{i:\mathbb{N}}$, named the cohomology groups, such that:

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- The invariants are supposed to be "easy" to compute, and "nice" groups : $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$
- We are working in synthetic mathematics, in HoTT where cohomology groups have a remarkably short definition :

$$H^i(X) := \| \, X \longrightarrow \|\mathbb{S}^i\|_i \, \|_0$$

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$\mathbb{S}^1$	$\mathbb{Z}$	$\mathbb{Z}$	1	1	1	1
$\mathbb{R}P^2$	$\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1
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Cohomology groups are not enough !

Cohomology groups are just invariant

- Some topological spaces are not isomorphic but they have the same cohomology groups

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The cup product and the comology ring

- There is a graded operation on the groups, the cup product

$$\smile : \; H^i(X) \longrightarrow H^j(X) \longrightarrow H^{i+j}(X)$$

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- Which enables to turn $H^*(X) := \bigoplus_{i:\mathbb{N}} H^i(X)$ in a ring named the cohomology ring
- This cohomology ring is <u>one more invariant</u> :

$$H^*(X) \ncong H^*(Y) \Longrightarrow X \ncong Y$$

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What do we need to solve ?

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- We want to be constructive

2. Building the direct sum and graded rings

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2.1 Adapting the classical direct sum

Adapting the classical direct sum

The classical direct sum

$$\bigoplus_{i:I} G_i := \{(g_i)_{i \in I} \mid \exists \text{ finite } J \subset I, \forall n \notin J, \quad g_n = 0 \in G_n\}$$

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A general definition ?

$$\sum_{f:\prod_{i:I}:G_i} \| \sum_{\substack{J:\mathrm{subset}(I)\\ J\ \mathrm{finite}}} \prod_{\substack{i:I\\ i\notin J}} f(i) \equiv 0_i \|$$

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A solution when I is $\mathbb{N}$

$$\bigoplus_{n:\mathbb{N}}^{\mathrm{Fun}} G_n \ := \ \sum_{f:\prod_{n:\mathbb{N}}:G_n} \| \sum_{k:\mathbb{N}} \prod_{\substack{k:\mathbb{N} \\ k<i}} f(i) \equiv 0_i \|$$

Building Graded Rings ?

Abelian group structure

Given $f, g : \bigoplus^{\text{Fun}}_{n:\mathbb{N}} G_n$, an abelian group structure can be defined pointwise:

$$(f + g)(n) = f(n) +_n g(n)$$

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A product ?

Given $\star : G_i \to G_j \to G_{i+j}$ over $(\mathbb{N}, 0, +)$, we would like to define :

$$(f \times g)(n) = \sum_{i=0}^{n} f(i) \star g(n-i)$$

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Transports are needed

$$(f \times g)(n) = \sum_{i=0}^{n} \uparrow_i^n (f(i) \star g(n-i))$$

Proving the properties ?

Proving associativity is however complicated, it unfolds to proving :

$$
\begin{aligned}
(f \times (g \times h))(n) &= \sum_{i=0}^{n} \uparrow_i^n (f(i) \star (g \times h)(n-i)) \\
&= \sum_{i=0}^{n} \uparrow_i^n \left(f(i) \star \left(\sum_{j=0}^{n-i} \uparrow_j^{n-i} (\ g(j) \star h(n-i-j)\ ) \right) \right) \\
&\equiv \ldots \\
&= \sum_{i=0}^{n} \uparrow_i^n \left(\left(\sum_{j=0}^{i} \uparrow_j^{i} (\ f(j) \star g(i-j)\ ) \right) \star h(n-i) \right) \\
&= \sum_{i=0}^{n} \uparrow_i^n ((f \times g)(i) \star h(n-i)) \\
&= ((f \times g) \times h)(n)
\end{aligned}
$$

2. Building the direct sum and graded rings

2.2 A quotient inductive type definition

A quotient inductive type definition

```
data ⊕HIT (I : Type) (G : I → AbGroup) : Type where
  -- Point constructors
  0⊕        : ⊕HIT I G
  base      : (n : I) → ⟨ G n ⟩ → ⊕HIT I G
  _+⊕_      : ⊕HIT I G → ⊕HIT I G → ⊕HIT I G
  -- Abelian group laws
  +⊕Assoc : ∀ x y z → x +⊕ (y +⊕ z) ≡ (x +⊕ y) +⊕ z
  +⊕Rid     : ∀ x → x +⊕ 0⊕ ≡ x
  +⊕Comm : ∀ x y → x +⊕ y ≡ y +⊕ x
  -- Morphism laws
  base0⊕    : ∀ n → base n 0⟨ G n ⟩ ≡ 0⊕
  base+⊕    : ∀ n x y → base n x +⊕ base n y ≡ base n (x +⟨ G n ⟩ y)
  -- Set truncation
  trunc     : isSet (⊕HIT I G)
```

Defining a graded ring

Defining the product

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- The different higher equations are easy to verify

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  base $(n+(m+k))\ \ (x\star(y\star z)) \ \equiv\ $ base $((n+m)+k)\ \ ((x\star y)\star z)$

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- All equations are trivial

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Multivariate polynomials

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- This enables a direct definition of multivariate polynomials
- The elements and the product are intuitive and easy to work with
  - ▷ Elements are generated by 0, aX^n, $+$
  - ▷ The product is basically generated by $aX^n \times bX^m = abX^{n+m}$

3. Proving the isomorphisms ?

A General Method

Objective ?

Prove ring isomorphisms of the form :

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3. Get a ring morphism $\mathbb{Z}[X,Y]/\langle X^2, XY, 2Y, Y^2\rangle \longrightarrow H^*(\mathbb{K}^2)$ by proving it cancels on $X^2, XY, 2Y, Y^2$

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3. Get a ring morphism $\mathbb{Z}[X,Y]/\langle X^2, XY, 2Y, Y^2\rangle \longrightarrow H^*(\mathbb{K}^2)$ by proving it cancels on $X^2, XY, 2Y, Y^2$
4. Build an inverse $H^*(\mathbb{K}^2) \longrightarrow \mathbb{Z}[X,Y]/\langle X^2, XY, 2Y, Y^2\rangle$ by recursion

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The method in practice

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A word on synthetic mathematics

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  - ▷ Possible to directly characterize the behavior of the cup product on spaces

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